

Aharonov-Bohm problem for spin-1

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ABSTRACT

The basic AB *problem* is to determine how an unshielded tube of magnetic flux Φ affects arbitrarily long-wavelength charged particles impinging on it. For spin-1 at almost all Φ the particles do not penetrate the tube, so the interaction essentially is periodic in Φ (AB *effect*). Below-threshold bound states move freely only along the tube axis, and consequent induced vacuum currents supplement rather than screen Φ . For a pure magnetic interaction the tube must be broader than the particle Compton wavelength, i.e., only the nonrelativistic spin-1 AB problem exists.

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I. Introduction

Aharonov and Bohm, in their first paper on the effect which has come to bear their name [1, 2], also introduced a novel problem in quantum physics. The AB *effect* is the set of measurable phenomena which occur for charged particles unable to penetrate an endless tube of magnetic flux – all observables are periodic in the flux (with period h/q , where q is the charge). What may properly be called the AB *problem* is determining the influence on charged particles of an unshielded flux tube, in the limit where the particle de Broglie wavelength goes to infinity. AB [1] observed that in this limit spin-0 particles do not penetrate the tube, so that conditions for the AB effect are satisfied automatically. Later work showed that there are no particle states bound inside the tube, and that the inability to penetrate still holds if the Compton wavelength is long compared to the de Broglie wavelength, i.e., the particle motion is relativistic. Further, there are induced vacuum currents outside the tube, generating an extra flux which screens the total flux towards the nearest integer number of flux quanta Nh/q [3].

For the spin- $\frac{1}{2}$ Dirac case there are interesting changes. Now particles are able to penetrate just enough to be sensitive to the sign of the flux [4]. This fact is connected with the existence of threshold bound states for electrons whose magnetic moment is aligned with the flux: If there are N whole quanta of flux, then in the $2+1$ dimensional problem obtained by factoring out motion in the direction along the tube there are N particle states (with magnetic moment parallel to the flux) confined inside the tube [5]. In the full $3+1$ dimensional problem, each such state corresponds to a distinguishable particle with exactly the free electron mass, and able to move only along the tube. If there is an additional fractional flux there is a ‘quasi-bound’ state, or equivalently a phase shift $\frac{\pi}{2}$ (with respect to the corresponding spin-0 case) at threshold for exactly one partial wave. The perfect ‘impedance match’ between the infinite-wavelength external wave and the internal state at exactly threshold energy is what permits this minimal nontrivial coupling between the flux and the outside particles beyond that implied by the AB effect. In the spin- $\frac{1}{2}$ case, induced external vacuum currents screen the magnitude of the flux down towards the nearest smaller integer [6], again showing dependence on the sign of the flux as well as its fractional part.

The aim of this work is to determine the corresponding answers to the AB problem for spin-1 Yang-Mills particles. We find that, except for a discrete set of flux values, there is no penetration by threshold-energy particles impinging on the tube. In this sense the situation resembles that for spin-0, where the AB effect holds exactly. However, now there is a set of below-threshold bound states, somewhat more numerous than the threshold bound states for spin- $\frac{1}{2}$. The most dramatic change is that, to have a pure magnetic field and no other forces affecting an incident particle, the tube must be broader than a vector boson Compton wavelength, so that there is no relativistic AB problem for spin-1. Finally, spin-1 vacuum currents enhance the given flux, a kind of anti-screening familiar from discussions of QCD and asymptotic freedom in the domain where only magnetic fields are considered [7]. Many qualitative and even quantitative results for the AB problem may be illuminated by the study of charged classical particles interacting with a narrow flux tube [8]: The reason is that the role of \hbar , the quantum of action, often may be played by another quantity with the same dimensions, $q\Phi$, the product of particle charge with magnetic flux.

The rest of the paper is organized as follows. In Sec. II we explain why the Yang-Mills equation is the appropriate analogue for spin-1 to the Klein-Gordon equation for spin-0 and the Dirac equation for spin- $\frac{1}{2}$. In Sec. III we discuss the scattering solutions for long wavelength, and find that the waves do not penetrate the flux, except for a discrete set of flux values. In Sec. IV we study the bound state solutions of the linearized equations, and find that the number of bound states is somewhat greater than the number of flux quanta. In Sec. V we analyze the behavior of the vacuum in the presence of a flux line, taking into account the crucial contributions to the classical Yang-Mills action quartic in the vector boson fields. In Sec. VI we find insignificant changes in the analysis if the magnetic field inside the tube is nonuniform.

II. Choice of equation

Developments of recent decades make the linearized Yang-Mills equation the obvious choice to describe electromagnetic interactions of charged vector bosons: By now the successes of the standard model for electroweak interactions and quantum chromodynamics for strong interactions show that nonabelian gauge invariance not only is

attractive esthetically but also is utilized by nature. Here lies the difference between our approach to the spin-1 problem and that of Hagen and Ramaswamy [9], who adopt the Proca equation, generalized by introducing minimal electromagnetic coupling into what originally was an equation for neutral vector bosons. The most general linear spin-1 equation [10] links the gyromagnetic ratio g to an electric quadrupole coupling proportional to $g - 2$. HR's assumptions give $g = 1$ and hence a nonzero quadrupole coupling, the origin of divergent high energy behavior which would preclude perturbative renormalizability [11]. Like the Dirac equation, the Yang-Mills equation implies $g = 2$, hence an exact lock between precession of spin and momentum in a uniform, static magnetic field. The YM choice manifests a symmetry of charged-particle motion in pure magnetic fields: In classical physics, particle trajectories depend on momentum, but not energy (which enters only in determining the speed at which any trajectory is traversed). The symmetry insures consistency between the spatial dependences of YM wave functions in the relativistic regime and in what HR call the Galilean regime, while for HR electric quadrupole coupling breaks this connection. Further, only for the YM case does the covariant divergence of the (four-vector) spin wave function vanish, sustaining the physical interpretation of the wave function as a purely spatial three-vector in the instantaneous rest frame of the charged particle.

HR's quadrupole coupling produces such pathological behavior in very strong magnetic fields that they require scattering functions not to penetrate the flux, making the relativistic AB problem trivial by fiat. We on the other hand find that very strong pure magnetic fields acting on charged vector bosons cannot occur, so that for physical reasons there is no relativistic AB problem. In what they call the Galilean limit, HR neglect the $\mathcal{O}(\frac{1}{M^2})$ electric quadrupole coupling, obtaining a well defined problem, but with $g = 1$ rather than the preferred value $g = 2$, and scattering resembling that for spin- $\frac{1}{2}$, instead of spin-0 as we find.

III. Linearized wave equation and threshold-energy scattering

In the Yang-Mills equations, the electromagnetic vector potential A_μ is identified with the $I_3 = 0$ part of the field, and the positively and negatively charged fields P_α and N_α are identified with the $I_3 = \pm 1$ parts, where I_3 is the third component of the

isospin. The equations may be written

$$[\mathcal{D}_\alpha, [\mathcal{D}^\alpha, \mathcal{D}_\beta]] = 0, \quad (1)$$

with

$$\begin{aligned} \mathcal{D}_\alpha &= \partial_\alpha - iqV_\alpha, \\ V_\alpha &= T_+P_\alpha + T_-N_\alpha + T_3A_\alpha \\ [T_i, T_j] &= i\epsilon_{ijk}T_k, \\ T_\pm &= T_1 \pm iT_2, \end{aligned} \quad (2)$$

and q the charge of the particle. Greek indices run over space and time; Roman over space only. From here on except where indicated explicitly, we use units with $\hbar = c = 1$. The positive charge projection (all terms with net unit positive charge) of (1) contains terms of the form AAP and PNP. The latter may be omitted to obtain an equation linear in the charged field.

For perturbative renormalizability the Higgs mechanism is needed to describe masses of vector bosons. In the linearized wave equation this is functionally equivalent to adding a term with a fixed mass M , so that the solutions P^α automatically obey the condition

$$D_\alpha P^\alpha = 0, \quad (3)$$

with $D_\alpha = \partial_\alpha - iqA_\alpha$ [10]. There occur in (1) two terms of the form $[D_\alpha, D_\beta]$. Recognizing $-i\epsilon_{ijk} = S_k$, one finds a magnetic moment interaction proportional to qsB (where $s = \pm 1, 0$ is the eigenvalue of $\mathbf{S} \cdot \hat{\mathbf{B}}$ acting on P). The resulting equation is

$$D_\beta D^\beta P^i + 2qsBP^i + M^2P^i = 0. \quad (4)$$

To solve (4), we choose the applied magnetic flux in the form of a uniform cylinder in the z -direction, with radius R taken to zero at the end of the calculation. Later we shall come back to the significance and generality of conclusions associated with assuming uniform field inside the tube. Because of the translational and boost symmetries in the z -direction we may restrict our analysis to the two transverse spatial dimensions.

The (external) kinetic energy is assumed small in comparison with the magnetic moment interaction inside the flux tube, and so is dropped. For a state localized well within the flux tube, the squared wave number $k^2 = E^2 - M^2$ is given by the following expression, in which the first term corresponds to the Landau level energy and the second corresponds to the magnetic moment interaction:

$$k^2 = \frac{4F}{R^2}(n + \frac{1}{2}) - \frac{2F}{R^2}gs. \quad (5)$$

Here the flux F is measured in units of an AB quantum of the conventional flux Φ , i.e., $F = q\Phi/2\pi$. For spin-1 particles with $g = 2$, this expression can be negative only for the lowest Landau level. Both inside and outside the flux tube the wave function may be expressed as

$$P(r, \phi) = e^{im\phi}f(r), \quad (6)$$

where $f(r)$ tacitly depends on the spin projection s and also on the integer azimuthal angular momentum m , which must be an integer for the wave function to be single-valued. Putting (6) and the cylindrical forms of the derivatives into (4) yields

$$f'' + \frac{1}{r}f' - \left[\left(\frac{m}{r} \right)^2 - qB(m + 2s) + \left(\frac{qBr}{2} \right)^2 \right] f = 0 \quad (7)$$

inside the flux cylinder, and

$$f'' + \frac{1}{r}f' - \left[\left(\frac{m}{r} \right)^2 - qBm \left(\frac{R}{r} \right)^2 + \left(\frac{qB}{2} \left(\frac{R^2}{r} \right) \right)^2 - k^2 \right] f = 0 \quad (8)$$

outside.

The exterior (Bessel) equation is independent of the spin. Its solution is:

$$f(r) = cJ_{|m-F|}(kr) + dY_{|m-F|}(kr), \quad (9)$$

where J and Y are respectively the regular and irregular Bessel functions, and again $F = qBR^2/2$ is the number of flux quanta. The interior solution may be approximated

by a series expansion,

$$f(r) = e^{-\frac{Fr^2}{2R^2}} \left(\frac{r}{R}\right)^{|m|} \times \quad (10)$$

$$\left(1 + \frac{\Gamma(1 + |m|)}{\Gamma(\frac{1}{2}(1 - m + |m| - 2s))} \sum_{j=1}^{\infty} \left[(F)^j \left(\frac{r}{R}\right)^{2j} \frac{\Gamma(j + \frac{1}{2}(1 - m + |m| - 2s))}{\Gamma(j + 1)\Gamma(j + |m| + 1)} \right] \right).$$

Note that this form is an asymptotic series, since the radial dependence of the coefficients in the differential equation precludes analyticity. Thus care is required in drawing quantitative conclusions from the use of this approximation, but it should be good enough for qualitative insight, as it exhibits the appropriate ‘antigaussian’ asymptotic behavior – growth at large r given by $e^{+qBr^2/4}$. In all the following, we shall insure that sufficient accuracy is available for the purposes at hand.

Now the inside and outside solutions must be matched at the flux boundary. The azimuthally dependent factors and their derivatives match already, so only radial matching conditions are needed. We use a two-step matching that simplifies the bookkeeping. Near the flux tube and for small enough values of its radius, the external solution may be written as

$$f(r) = a \left(\frac{r}{R}\right)^{|m-F|} + b \left(\frac{r}{R}\right)^{-|m-F|}. \quad (11)$$

The relationship between the coefficients in (9) and those in (11) is obtained by expanding (9) (using standard asymptotic formulae for Bessel functions of small argument [12]) and setting this equal to (11). At the boundary R , the dimensionless quantities

$$\mathbf{D} = R \frac{f'}{f} \quad (12)$$

for (10), and for (11) must match. This means c/d must satisfy the equation

$$\frac{|m - F| + \mathbf{D}}{|m - F| - \mathbf{D}} = - \left(\frac{kR}{2}\right)^{2|m-F|} \frac{\Gamma(1 - |m - F|) \left(c + d \frac{\cos(|m-F|\pi)}{\sin(|m-F|\pi)}\right)}{\Gamma(1 + |m - F|) \left(d \frac{1}{\sin(|m-F|\pi)}\right)}. \quad (13)$$

The relationship between c and d determines the behavior of the wave function at large

values of the argument (kr). For $k \geq 0$, the phase shift is defined by

$$\tan(\delta) = -d/c. \quad (14)$$

Note that we define δ in such a way that it would vanish if the charged particle were excluded from the flux tube. Of course there is still an AB centrifugal potential, which means that there is a phase shift from the case of no flux, but that effect is well-understood; it is the possibility of deviations from the pure AB case which we are trying to address here. The behavior of δ as a function of F is given by (13). For less than critical values of the flux, δ is small and positive. At the critical value of F , δ rises sharply through $\pi/2$ to just below π , where it remains for larger than critical flux. The size of kR determines how sharp the transition is. For $kR = 0$, $\delta(F)$ is a step function. The transition from a free to a bound state occurs when (13) can be satisfied for $k = 0$. At precisely this value of F , a quasi-bound state exists, i.e., d/c diverges as k approaches zero. Such a wave function has an infinitely long tail, so that it is not square integrable, but for infinitesimally larger F it would be a true bound state. According to (13) the quasi-bound state occurs for F such that

$$|m - F| + \mathbf{D} = 0, \quad (15)$$

of course only possible when the magnetic moment and the flux are parallel. The quasi-bound state with the smallest flux occurs for $m = 1$ at $F = 0.74$. Note that the seeming solution of (15) $F = 0$ is spurious. Instead, for $m = 0$ and any $F \neq 0$ there is a true bound state, the more deeply bound the bigger $|F|$ is.

Just as in the case of spin- $\frac{1}{2}$ (and in HR's Galilean limit of the Proca scheme for spin-1), where quasi-bound states exist for all noninteger F , the existence of such a state implies penetration of the flux tube by the particle, sufficient to produce sensitivity to the sign of the flux. The difference for spin-1 Yang-Mills particles is that the quasi-bound states exist only for discrete values of the flux, so that penetration occurs only for flux values in a set of measure zero.

IV. Counting bound states

At energies less than the mass, i.e., $k^2 < 0$, the matching conditions yield (13) with k replaced by $i\kappa$. Since these are bound states and not just quasi-bound, the large r behavior must be a decaying exponential, which means $c/d = 1/i$. Then F must satisfy

$$\frac{|m - F| + \mathbf{D}}{|m - F| - \mathbf{D}} = -\left(\frac{\kappa R}{2}\right)^{2|m-F|} \frac{\Gamma(1 - |m - F|)}{\Gamma(1 + |m - F|)}. \quad (16)$$

For $\kappa = 0$, this also dictates that F satisfy (15): One has approached the quasi-bound-state limit from the bound-state rather than the scattering side, but the limiting behavior is the same. Therefore a value of F greater than a critical value by even the smallest amount implies the existence of a true bound state in the corresponding partial wave. As mentioned earlier, despite the fact that for $m = 0$ there is never a quasi-bound state, a true bound state does exist for any non-zero value of F . To count the total number of bound states we need to find that value of F for which a quasi-bound state appears at a given m ; any F slightly greater than this yields exactly $m + 1$ bound states.

The dependence of the total number of bound states on F can be inferred at least roughly from the approximate solutions of (15). For each increase of m by one, the number of possible bound states increases by one. Therefore, the change in F per added bound state at some value of F can be found by solving (15) for pairs of adjacent values of m . We fit a curve to points obtained this way, using the approximation (10), and sought to obtain an asymptotic form for $\frac{dm}{dF}$. Integrating the resulting expression gave an estimate for the number of bound states ν as a function of the amount of flux,

$$\nu \approx F + 0.3\sqrt{F}. \quad (17)$$

Because we know that the series method is not quantitatively reliable, this result needs further examination. First, it is worth noting that the qualitative character of (17) is quite reasonable. Since states in the lowest Landau level all are bound, there should be at least $[F + 1]$ of them. Near the edge of the tube, there should be extra room for some less strongly bound configurations, and the number of these should be proportional to the circumference of the tube $2\pi R$, measured on the scale of the magnetic length, $R\sqrt{\pi/F}$. Thus simple geometry underlies this extra contribution to the number of bound states.

In the large F limit an asymptotic form for m_{max} as a function of F (where m_{max} is the maximum azimuthal quantum number corresponding to a bound state) can be found by writing (4) in the form

$$\frac{d^2 f(r)}{dr^2} + \frac{1}{r} \frac{df(r)}{dr} + k^2(r)f(r) = 0, \quad (18)$$

where

$$k^2(r) = \frac{4F}{R^2} - \left(\frac{m - F \left(\frac{r}{R} \right)^2}{r} \right)^2. \quad (19)$$

We expand $k(r \equiv \bar{R} - \delta)$ through second order about its minimum at $r = \bar{R} = R\sqrt{m/F}$ (even though we know $\bar{R} > R$, the outer radius of the tube), and make the substitution

$$x = \sqrt{F}\delta/R. \quad (20)$$

If we assume that m can be written as

$$m = F + \alpha\sqrt{F}, \quad (21)$$

then, recalling the assumption that F is very large, (18) becomes

$$\frac{d^2 \tilde{f}(x)}{dx^2} + 4(1 - x^2)\tilde{f}(x) = 0. \quad (22)$$

Matching logarithmic derivatives across the flux tube boundary results in an equation for α :

$$\frac{\tilde{f}'\left(\frac{\alpha}{2}\right)}{\tilde{f}\left(\frac{\alpha}{2}\right)} = \alpha, \quad (23)$$

where the radius R of the flux tube corresponds to $x = \alpha/2$. A direct numerical solution of (22) converged well and gave

$$\alpha = 0.55. \quad (24)$$

To this same two-place accuracy, the JWKB approximation carried consistently through second order in x gives the same result, which is rather impressive, as the inside-outside matching condition is imposed not far from the classical turning point where the approximation has a spurious square root divergence.

V. Vacuum polarization effects

Having counted the bound state solutions to the linear wave equation, we need to analyze their effect on the vacuum structure. If the flux is spread out on a scale large compared to the boson Compton wavelength, then the bound states have positive energy smaller than the rest mass. Clearly this lowers the vacuum energy compared to that in the absence of the flux, and therefore produces a paramagnetic effect enhancing that flux, a clear example of antiscreening. The antiscreening may compete with, but should dominate, the effect of threshold scattering states, which as for spin-0 tend to bring the flux to the nearest quantum value, whether larger or smaller in magnitude.

A quite different situation arises if the flux is assumed to be concentrated so that the magnetic length is less than the Compton wavelength. In this case, the bound states have $\omega^2 = -\kappa^2 + m^2 < 0$, so that the frequency is imaginary, and the bound state amplitudes grow exponentially with time. This is not vacuum polarization, but rather instability of what one would naively identify as the vacuum. The first thing one can say is that this instability must be halted by the terms in the energy quartic in the charged boson field, which act as an effective mass proportional to the field amplitude, and eventually must counterbalance the negative quadratic terms responsible for the instability. It is an interesting question worth further study whether the configuration obtained by optimizing the coefficients of the unstable modes of the linearized equation is itself stable, or whether additional instabilities bring about the complete extinction of the entire Yang-Mills field strength inside a very narrow tube.

There are several reasons to believe that this might be the case. First, on distance scales small compared to the boson Compton wavelength the full nonabelian gauge invariance is manifest, and the flux, which is a gauge covariant rather than invariant quantity, should not be a physical observable with a definite nonzero expectation value. Secondly, if we try to imagine how this flux could be created, it would require a

cylindrical sheet of intense current. The gauge interaction of the particles producing this current would generate huge quantum fluctuations in the isospin orientation of each particle, so that its charge would average to zero, as would the corresponding current. Hence there would be no steady source for the flux, and so no flux. Finally, Nielsen and Olesen [13] observed that a vacuum instability in QCD which favors formation of a uniform nonzero magnetic field does not by itself end in a stable configuration. There is a further instability to formation of what they call flux spaghetti, i.e., a complicated pattern of tubes of flux rapidly varying in space and time. This suggests that a single isolated flux tube of very small radius cannot occur. Either there are none, or there are many tending to cancel each other. Here we find the most dramatic change from the situations for lower spin. There is nothing inconsistent about a flux line influencing spin-0 or spin- $\frac{1}{2}$. However, for spin-1, the unadorned flux-line concept only makes sense if the particles are nonrelativistic, so that the flux tube size can be bigger than the boson Compton wavelength, yet still smaller than the de Broglie wavelength.

The above conclusion is consistent with known models for flux tubes in relativistic field theories. In these models, the tubes are examples of cosmic strings, with finite energy per unit length. The radius of such a string is determined by a force balance which automatically precludes magnetic lengths smaller than the vector boson Compton wavelength [14].

VI. Nonuniform field distributions

We promised to consider cases where the magnetic field is not uniform inside the tube. A nonuniformity involving magnetic length scales smaller than the Compton wavelength appears unphysical, for the reasons just discussed. Otherwise, the conclusion for the uniform-field case should continue to hold, that except for flux configurations in a set of measure zero where quasi-bound states occur, the scattering solutions at large de Broglie wavelength do not penetrate the flux. For bound states, the situation could be more complicated. For example, suppose that there were many ‘islands’ of flux, each carrying a positive flux $F_i < 0.74$. Provided there were sufficient spacing between islands compared to the radius of any one, each island would have one bound state, and the total number of bound states for large total F would be proportional to

F but with a proportionality constant $1/F_i > 1$. If field of both signs is allowed, then the number of bound states N could exceed the net flux F by an arbitrarily large factor, but the difference between the numbers of spin-up and of spin-down bound states would be more closely linked to F . This statement actually applies also to the spin- $\frac{1}{2}$ case, where there is an exact index theorem, $N_{up} - N_{down} = [F]$ [5]. There the exterior behavior, i.e., the finite-energy scattering, depends only on the sign of the total flux and on its fractional part.

For all three spins the low-energy scattering on a flux tube is determined by the fractional part of the flux, and for spin- $\frac{1}{2}$ also the sign. For spin-0 there are no bound states regardless of the distribution of the flux, but for the higher spins the number of bound states is sensitive to the distribution. Thus one finds the unsurprising conclusion that behavior inside the flux tube depends on the distribution, but behavior outside is completely unaffected, except for configurations in a set of measure zero in the case of spin-1. In other words, exterior sensitivity to the flux distribution, as opposed to the total flux, shows little or no change with spin, precisely because there is little or no penetration of the flux.

VII. Conclusions – Spin metamorphoses of the Aharonov-Bohm problem

The problem of a charged particle in the presence of a flux line originated with the paper of Aharonov and Bohm [1], where they observed that in the absence of spin the particle automatically is excluded from the flux. Thus all phenomena must be periodic in the flux, with a period of one AB flux quantum. For spin- $\frac{1}{2}$ there are exactly $[F]$ normalizable zero-energy states bound inside the flux, and also one quasi-bound state, as long as F exceeds its integer part $[F]$ by any nonzero amount [5]. It is this feature which allows the wave function to penetrate the flux just enough to be sensitive to its sign, thus slightly spoiling the perfect AB periodicity of the spinless case, and violating usual expectations for decoupling between phenomena at very different scales. Nevertheless, the problem of spin- $\frac{1}{2}$ particles interacting with an arbitrarily thin flux tube remains well-defined. For spin-1, the linearized Yang-Mills equation has clear solutions in the limit of zero tube radius, but the bound state solutions growing with time (which appear if the limit is taken on the scale of the Compton wavelength of the

vector boson) are physically unacceptable. Fortunately, the nonlinearities in the Yang-Mills system conspire to make this limit unachievable. On length scales large compared to the Compton wavelength the limit does make sense, and the description of scattering and the counting of bound states go through exactly as described in the body of the paper. Such a ‘fat’ flux line would polarize the vacuum so as to enhance the applied flux. As the tube radius cannot be made small compared to the Compton wavelength, the spin-1 case appears to be the end of the road for the relativistic Aharonov-Bohm problem, though the more complex problem which replaces it deserves further study.

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